Multi-domain spectral methods for the computation of fractional derivatives

Fractional derivatives have seen in recent years an increase in importance in applications since they allow a simple modelling of nonlocal effects, see for instance [1, 2, 3] and references therein. Fractional derivatives of order s, $0 < s < 1$ are most easily defined via their Fourier transform

$$\widehat{\partial_x^s u} = |\xi|^s \hat{u},$$

where $\hat{u}$ is the Fourier transform of u and where ξ is the Fourier variable dual to x. Alternatively they can be defined as

$$\partial_x^s u \propto \int_{-\infty}^{x} \frac{u(y)}{(x-y)^s} dy - \int_{\infty}^{x} \frac{u(y)}{(x-y)^s} dy.$$

The problem in the numerical computation of fractional derivatives is the slow algebraic decrease of the fundamental solutions of equations as in [1] which makes the use of Fourier methods inefficient.

The task is to write a code to implement a multidomain spectral method to compute the above integrals. The idea is to separate the real line into several intervals which are all mapped to $[-1, 1]$. On each of them a substitution is performed to obtain a smooth integrand on the respective interval. The integrals are computed via Clenshaw-Curtis integration, see e.g. [4], which is both efficient and of spectral accuracy. The resulting code will be applied to the numerical construction of the soliton solutions of [1].

References

[1] R.L. Frank and E. Lenzmann, *On the uniqueness and nondegeneracy of ground states of* $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ *in* $\mathbb{R}$, Acta Math. **210** (2) (2013), 261-318.

[2] C. Klein and J.-C. Saut, *A numerical approach to blow-up issues for dispersive perturbations of Burgers' equation*, Physica D: Nonlinear Phenomena (2015), 46-65, 10.1016/j.physd.2014.12.004

[3] C. Klein, C. Sparber and P. Markowich, *Numerical study of fractional Nonlinear Schrödinger equations*, Proc. R. Soc. A 470: (2014) DOI: 10.1098/rspa.2014.0364.

[4] L.N. Trefethen, Spectral Methods in Matlab. SIAM, Philadelphia, PA (2000)