Newton-Krylov iterative approach for stiff partial differential equations

Many partial differential equations (PDEs) appearing in hydrodynamics and nonlinear optics have higher order derivatives in the nonlinear part. The most prominent of these might be the Camassa-Holm (CH) equation [1]

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

with k a real constant, an integrable generalization of the Korteweg-de Vries equation obtained via asymptotic expansions around simple wave motion of the one-dimensional Euler equations for shallow water. A whole class of PDEs with a similar property was given by Dubrovin, see for instance [2], where solutions of such equations were studied numerically. The technical problem in this context is that the systems of ordinary differential equations (ODEs) obtained after a discretization of the spatial coordinates are *stiff*. This means that vastly different time scales have to be resolved numerically which makes the use of explicit integration schemes inefficient. Implicit methods with a fix point iteration are stable, but might fail to converge near critical points of the solutions as in [2].

An improvement of the approach would be to use a Newton iteration instead. Because of the high spatial resolution needed in this context, the inversion of the Jacobian cannot be computed directly on serial computers. Krylov subspace methods as GMRES [3] allow an iterative approach to the inversion of the Jacobian called a Newton-Krylov method. The task is to write a code in Matlab to solve the CH equation numerically by using a discrete Fourier transform for the spatial coordinates and an implicit Runge-Kutta method of 4th order of the time dependence. Since in GMRES only the action of the to be inverted matrix on a vector has to be known, the Fourier approach can use the efficient fast Fourier transform.

References

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[3] Y. Saad and M. Schultz, *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems.* SIAM J. Sci. Comput. **7** (1986), no. 3, 856–869.